

## Spectral Stability for Subsonic Traveling Pulses of the Boussinesq “abc” System\*

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**Abstract.** We consider the spectral stability of certain traveling wave solutions for the Boussinesq “abc” system. More precisely, we consider the explicit  $\text{sech}^2(x)$  like solutions of the form  $(\varphi(x - wt), \psi(x - wt)) = (\varphi, \text{const.}\varphi)$ , exhibited by M. Chen [*Appl. Math. Lett.*, 11 (1998), pp. 45–49], [*Internat. J. Theoret. Phys.*, 37, (1998), pp. 1547–1567], and we provide a complete rigorous characterization of the spectral stability in all cases for which  $a = c < 0$ ,  $b > 0$ .

**Key words.** linear stability, traveling waves, Boussinesq system

**AMS subject classifications.** 35B35, 35B40, 35G30

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### 1. Introduction and results.

**1.1. The general Boussinesq “abcd” model.** In this work, we are concerned with the Boussinesq system

$$(1) \quad \begin{cases} \eta_t + u_x + (\eta u)_x + a u_{xxx} - b \eta_{xxt} = 0, \\ u_t + \eta_x + u u_x + c \eta_{xx} - d u_{xxt} = 0. \end{cases}$$

The first formal derivation for this system has appeared in the work of Bona, Chen, and Saut [5] to describe the (essentially two dimensional) motion of small-amplitude long waves on the surface of an ideal fluid under the force of gravity. Here,  $\eta$  represents the vertical deviation of the free surface from its rest position, while  $u$  is the horizontal velocity at time  $t$ . In the case of zero surface tension  $\tau = 0$ , the constants  $a, b, c, d$  must additionally satisfy the consistency conditions  $a + b = \frac{1}{2}(\theta^2 - 1/3)$  and  $c + d = \frac{1}{2}(1 - \theta^2) > 0$ . In the case of nonzero surface tension, however, one requires only  $a + b + c + d = \frac{1}{3} - \tau$ . For this reason (as well as from the pure mathematical interest in the analysis of (1)), one may as well consider (1) for all values of the parameters.

Systems of the form (1) have been the subject of intensive investigation over the last decade. In particular, the role of the parameters  $a, b, c, d$  in the actual fluid models has been explored in great detail in the original paper [5] and later in [6]. It was argued that only

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models in the form (1) for which one has linear and nonlinear well-posedness are physically relevant. We refer the reader to these two papers for further discussion and some precise conditions under which one has such well-posedness theorems.

Regarding explicit traveling wave solutions, Chen has considered various cases of interest in [7], [8]. In fact, she has written numerous traveling wave solutions (i.e., in the form  $(\eta, u) = (\varphi(x - wt), \psi(x - wt))$ ); in fact, some of them are not necessarily homoclinic to zero at  $\pm\infty$ . In a subsequent paper [9] Chen also found new and explicit multipulsed traveling wave solutions.

In [11], Chen, Chen, and Nguyen consider another relevant case, namely the Benjamin–Bona–Mahoney (BBM) system ( $a = c = 0$ ,  $b = d = \frac{1}{6}$ ). They construct periodic traveling wave solutions for the BBM case as well as in more general situations. In [2], the authors explore the existence theory for the the BBM system as well as its relations to the single BBM equation.

We wish to discuss another aspect of (1), which is its Hamiltonian formulation. Since it is derived from the Euler equation by ignoring the effects of the dissipation, one generally expects such systems to exhibit a Hamiltonian structure. *This is however not generally the case, unless one imposes some further restrictions on the parameters.* Indeed, if  $b = d$ , one can easily check that

$$H(\eta, u) = \int -c\eta_x^2 - au_x^2 + \eta^2 + (1 + \eta)u^2 dx.$$

Furthermore,  $H(\eta, u)$  is positive definite only if  $a, c < 0$ . From this point of view, it looks natural to consider the case  $b = d$  and  $a, c < 0$ . In order to focus our discussion, we shall concentrate then on this version:

$$(2) \quad \begin{cases} \eta_t + u_x + (\eta u)_x + au_{xxx} - b\eta_{xxt} = 0, \\ u_t + \eta_x + uu_x + c\eta_{xxx} - bu_{xxt} = 0. \end{cases}$$

We will refer to (2) as the Boussinesq “abc” system. It is a standard practice that stable coherent structures, such as traveling pulses, are produced as constrained minimizers of the corresponding (positive definite) Hamiltonians, with respect to a fixed conserved quantity. In fact, this program has been mostly carried out, at least in the Hamiltonian cases, in a series of papers by Chen, Nguyen, and Sun. More precisely, in [12], the authors have shown that traveling waves for (1) exist in the regime<sup>1</sup>  $b = d$ ,  $a, c < 0$ ,  $ac > b^2$ . In addition, they have also shown stability of such waves in the sense of a “set stability” of the set of minimizers. In the companion paper [13], the authors have considered the general case  $b = d > 0$ ,  $a, c < 0$ , which in particular allows for small surface tension.

The existence of a traveling wave was proved for every speed  $|w| \in (0, \min(1, \frac{\sqrt{ac}}{b}))$ . This is the so-called subsonic regime. Finally, we point out a recent work by Chen et al. [10] in which the authors study numerically various aspects of spectral stability/instability of some solitary waves of (1), including the multipulsed solutions exhibited in [9]. In the same paper, the authors also study (numerically) the transverse stability/instability of the same waves, viewed as solutions to the two dimensional problem.

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<sup>1</sup>Which in particular requires that  $a + b + c + d < 0$ , corresponding to a “large” surface tension  $\tau > \frac{1}{3}$ .

The purpose of this paper is to rigorously study the spectral stability of some explicit traveling waves in the regime  $b = d > 0$ ,  $a, c < 0$ . This would be achieved via the use of the instability indices counting formulas of Kapitula, Kevrekidis, and Sandstede [15], [16] and the subsequent refinement by Kapitula and Stefanov [17].

**1.2. The traveling wave solutions.** In this section, we follow almost verbatim the description of some explicit solutions of interest of (1), given by Chen [7]; see also the more detailed exposition of the same results in [8]. More precisely, the solutions of interest are traveling waves, that is, in the form

$$\eta = \varphi(x - wt), \quad u(x, t) = \psi(x - wt).$$

A direct computation shows that if we require that the pair  $(\varphi, \psi)$  vanish at  $\pm\infty$ , then it satisfies the system

$$(3) \quad \begin{cases} (1 + c\partial_x^2)\varphi - w(1 - b\partial_x^2)\psi + \frac{\psi^2}{2} = 0, \\ -w(1 - b\partial_x^2)\varphi + (1 + a\partial_x^2)\psi + \varphi\psi = 0. \end{cases}$$

The typical ansatz that one starts with, in order to simplify the system (3) to a single equation, is  $\psi = B\varphi$ . This has been worked out by Chen [7], [8]. The following result is contained in the said papers.

**Theorem 1 (Chen, [7], [8]).** *Let the parameters  $a, b, c$  in the system satisfy one of the following:*

- (1)  $a + b \neq 0$ ,  $p = \frac{c+b}{a+b} > 0$ ,  $(p - 1/2)((b - a)p - b) > 0$ .
- (2)  $a = c = -b$ ,  $b > 0$ .

*Then, there are the (pair of) exact traveling wave solutions (i.e., solutions of (3))  $(\varphi(x - wt), \psi(x - wt))$ , where*

$$\begin{aligned} \varphi(x) &= \eta_0 \operatorname{sech}^2(\lambda x), \\ \psi(x) &= B(\eta_0)\eta_0 \operatorname{sech}^2(\lambda x), \end{aligned}$$

and

$$w = w(\eta_0) = \pm \frac{3 + 2\eta_0}{\sqrt{3(3 + \eta_0)}}, \quad \lambda = \frac{1}{2} \sqrt{\frac{2\eta_0}{3(a - b) + 2b(\eta_0 + 3)}}, \quad B(\eta_0) = \pm \sqrt{\frac{3}{\eta_0 + 3}},$$

and  $\eta_0$  is a constant that satisfies

- (1)  $\eta_0 = \frac{3(1-2p)}{2p}$  in case (1), and
- (2)  $\eta_0 > -3$ ,  $\eta_0 \neq 0$ , in case (2).

**1.3. Different notions of stability.** Before we state our results, we pause to discuss the various definitions of stability. First, one says that the solitary wave solution  $(\varphi_w, \psi_w)$  is orbitally stable if, for every  $\varepsilon > 0$ , there exists  $\delta > 0$ , so that whenever  $\|(f, g) - (\varphi_w, \psi_w)\|_X < \delta$ , one has that the corresponding solutions  $(\eta, u) : (f, g) = (\eta, u)|_{t=0}$ ,

$$\sup_{t>0} \inf_{x_0} \|(\eta(x - x_0, t), u(x - x_0, t)) - (\varphi(x - wt), \psi(x - wt))\|_X < \varepsilon.$$

Note that we have not quite specified a space  $X$ , since this usually depends on the particular problem at hand (and mostly on the available conserved quantities), but it suffices to say that  $X$  is usually chosen to be a natural energy space for the problem. This notion of (nonlinear) stability has been of course successfully used to treat a great deal of important problems, due to the versatility of the classical Benjamin and Grillakis–Shatah–Strauss (GSS) approaches. However, it looks like these methods are not readily applicable (if at all) to the Boussinesq abc system.

First, it is not clear whether the solutions described in Theorem 1 are in fact minimizers of the corresponding constrained variational problem. On the other hand, because of the conservation of momentum

$$I(\eta, u) = \int (\eta u + \eta_x u_x) dx,$$

it is quite possible that the GSS-type analysis will yield orbital stability for the actual minimizers. Note, however, that this is only potentially possible for the solutions outlined in case (2) if they are indeed minimizers (due to the presence of the continuous parameter  $\eta_0$ ). In case (1), due to the presence of the *discrete parameter*  $\eta_0$ , it does not seem that the GSS theory is applicable (even if these waves are actual minimizers), since one would need to compute the sign of the second derivative of the GSS function  $d$ , which is known only at discrete values of the parameter space.

We encourage the interested reader to consult the discussion in [12] (which contains the relevant conserved quantities but not much in terms of a discussion on why GSS fails), where a weaker but related stability was established in the regime  $ac > b^2$  and an additional smallness assumption on the wave is required as well. This is why one needs to develop an alternative approach to this important problem, which is one of the main goals of this work.

In this paper, we will concentrate on spectral stability. There is also (the closely related and almost equivalent) notion of linear stability, which we also mention below. In order to introduce the object of our study, as well as motivate its relevance, let us perform a linearization of the nonlinear system (2). Using the ansatz

$$\begin{cases} \eta = \varphi(x - wt) + v(t, x - wt), \\ u = \psi(x - wt) + z(t, x - wt) \end{cases}$$

in (2) and ignoring all quadratic terms in the form  $O(v^2)$ ,  $O(vz)$ ,  $O(z^2)$  leads to the following linearized problem:

$$(1 - b\partial_x^2) \begin{pmatrix} v \\ z \end{pmatrix}_t = -\partial_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 + c\partial_x^2 & bw\partial_x^2 + \psi - w \\ bw\partial_x^2 + \psi - w & 1 + a\partial_x^2 + \varphi \end{pmatrix}.$$

Letting

$$(4) \quad L = \begin{pmatrix} 1 + c\partial_x^2 & bw\partial_x^2 + \psi - w \\ bw\partial_x^2 + \psi - w & 1 + a\partial_x^2 + \varphi \end{pmatrix}, \quad J = -\partial_x(1 - b\partial_x^2)^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

the linearized problem that we need to consider may be written in the form

$$(5) \quad u_t = JLu.$$

Note that in the whole line context,  $L$  is a self-adjoint operator when considered with the natural domain  $D(L) = H^2(\mathbf{R}^1) \times H^2(\mathbf{R}^1)$ . Letting  $H := JL$ , we see that the problem (5) is in the form  $u_t = Hu$ . The study of linear problems in this form is at the basis of the deep theory of  $C_0$  semigroups. Informally, if the Cauchy problem  $u_t = Hu$  has global solutions for all smooth and decaying data, we say that  $H$  generates a  $C_0$  semigroup  $\{T(t)\}_{t>0}$  via the exponential map  $T(t) = e^{tH}$ . Furthermore, we say that we have *linear stability* for the linearized problem  $u_t = Hu$  whenever the growth rate of the semigroup is zero or equivalently  $\lim_{t \rightarrow \infty} e^{-\delta t} \|T(t)f\| = 0$  for all  $\delta > 0$  and for all sufficiently smooth and decaying functions  $f$ . Finally, we say that the system is *spectrally stable* if  $\sigma(H) \subset \{z : \Re z \leq 0\}$ . It is well known that if  $H$  generates a  $C_0$  semigroup, then linear stability implies spectral stability, but not vice versa. Nevertheless, the two notions are very closely related, and in many cases (including the ones under consideration), they are indeed equivalent. For the purposes of a formal definition, we proceed as follows.

**Definition 1.** We say that the problem (5) is unstable if there are  $\mathbf{f} \in H^2(\mathbf{R}^1) \times H^2(\mathbf{R}^1)$  and  $\lambda : \Re \lambda > 0$  so that

$$(6) \quad JL\mathbf{f} = \lambda\mathbf{f}.$$

Otherwise, the problem (5) is stable. That is, stability is equivalent to the absence of solutions of (6) with  $\lambda : \Re \lambda > 0$ .

**1.4. Main results.** We are now ready to state our results. We chose to split them into two cases, just as in Theorem 1. For the case  $a = c = -b$ ,  $b > 0$ , we have the following theorem.

**Theorem 2.** Let  $a = c = -b$ ,  $b > 0$ . Then, the traveling wave solutions of the abc system

$$(7) \quad \left( \eta_0 \operatorname{sech}^2 \left( \frac{x - wt}{2\sqrt{b}} \right), \pm \eta_0 \sqrt{\frac{3}{\eta_0 + 3}} \operatorname{sech}^2 \left( \frac{x - wt}{2\sqrt{b}} \right) \right)$$

with speed  $w = \pm \frac{3+2\eta_0}{\sqrt{3(3+\eta_0)}}$  are stable for all  $\eta_0 : \eta_0 \in (-\frac{9}{4}, 0)$ . Equivalently, all waves in (7) are stable for all subsonic speeds  $|w| < 1$ .

Note that  $|w| < 1$  is equivalent to  $\eta_0 \in (-\frac{9}{4}, 0)$ , so we assume this henceforth. Note that while supersonic waves do exist (these are all waves corresponding to  $\eta_0 \in (-3, -9/4) \cup (0, \infty)$  in Theorem 1), we cannot handle this case due to the dramatic failure of the index counting theories. That is because condition (2) of Theorem 4 fails to hold true. Indeed, in the case of supersonic waves one needs to deal with linearized operators of the form  $JL$ , where

$$(8) \quad L = \begin{pmatrix} L_1 & W \\ W & -L_2 \end{pmatrix},$$

where  $W$  is a bounded symmetric operator and  $L_1, L_2$  are self-adjoint operators, with  $P_{a.c.}(L_1)L_1 > 0$ ,  $P_{a.c.}(L_2)L_2 > 0$ . For a concrete manifestation of this phenomenon, we refer the reader to the equivalent formulation (17) of the eigenvalue problem at hand. There, it can be clearly seen that if  $|w| > 1$ , then one has to deal with a situation like (8).

In the remaining case, we assume only  $a = c < 0$ ,  $b = d > 0$ , but we observe that in this case Theorem 1 requires that  $\eta_0 = -3/2$ ,  $w = 0$ , that is, that the traveling waves become standing waves.

**Theorem 3.** Let  $a = c < 0$ ,  $b = d > 0$ . Then, the standing wave solutions of the Boussinesq system

$$\varphi(x) = -\frac{3}{2} \operatorname{sech}^2\left(\frac{x}{2\sqrt{-a}}\right), \quad \psi(x) = \pm \frac{3}{\sqrt{2}} \operatorname{sech}^2\left(\frac{x}{2\sqrt{-a}}\right)$$

are spectrally stable if and only if

$$(9) \quad \langle (a\partial_x^2 + 1 - \varphi)^{-1}(\varphi - b\varphi''), (\varphi - b\varphi'') \rangle \leq 8\sqrt{-a} \left( \frac{9}{2} + \frac{12}{5} \frac{b}{|a|} - \frac{3}{10} \frac{b^2}{a^2} \right).$$

Furthermore, there exists an absolute constant  $\gamma$  so that the condition (9) is satisfied (and hence the waves are stable) if and only if

$$0 < \frac{b}{|a|} < \gamma \sim 8.42083.$$

That is, the waves are unstable in the complementary region  $\frac{b}{|a|} > \gamma$ . The numerical value of  $\gamma$  is obtained by means of numerical simulations.

**2. Preliminaries.** In this section, we collect some preliminary results, which will be useful in what follows.

**2.1. Some spectral properties of  $L$ .** We shall need some spectral information about the operator  $L$ . We collect the results in the following proposition.

**Proposition 1.** Let  $a, c < 0$  and  $w : 0 \leq |w| < \min(1, \frac{\sqrt{ac}}{|b|})$ . Then, the self-adjoint operator  $L$  has the following spectral properties:

- The operator  $L$  has an eigenvalue at zero, with an eigenvector  $(\varphi', \psi')$ .
- There is  $\kappa > 0$  so that the essential spectrum is in  $\sigma_{\text{ess}}(L) \subset [\kappa, \infty)$ .

*Proof.* The first property is easy to establish; this is the usual eigenvalue at zero generated by translational invariance. For the proof, all one needs to do is take a spatial derivative in the defining system (3), whence  $L(\begin{smallmatrix} \varphi' \\ \psi' \end{smallmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .

Regarding the essential spectrum, we reduce matters to the Weyl theorem (using the vanishing of the waves at  $\pm\infty$ ), which ensures that

$$\sigma_{\text{ess.}}(L) = \sigma_{\text{ess.}}(L_0) = \sigma \left[ \begin{pmatrix} 1 + c\partial_x^2 & bw\partial_x^2 - w \\ bw\partial_x^2 - w & 1 + a\partial_x^2 \end{pmatrix} \right].$$

That is, it remains to check that the matrix differential operator  $L_0 > \kappa$ . By Fourier transforming  $L_0$ , it will suffice to check that the matrix

$$L_0(\xi) = \begin{pmatrix} 1 - c\xi^2 & -w(b\xi^2 + 1) \\ -w(b\xi^2 + 1) & 1 - a\xi^2 \end{pmatrix}$$

is positive definite for all  $\xi \in \mathbf{R}^1$ . Since  $1 - c\xi^2 \geq 1$ , it will suffice to check that the determinant has a positive minimum over  $\xi \in \mathbf{R}^1$ . We have

$$\det(L_0(\xi)) = \xi^4(ac - b^2w^2) + \xi^2(-a - c - 2bw^2) + (1 - w^2) \geq (1 - w^2) + 2\xi^2(\sqrt{ac} - |b|w^2),$$

where in the last inequality we have used  $-a - c \geq 2\sqrt{ac}$ . The strict positivity follows by observing that  $\sqrt{ac} \geq |b|w \geq |b|w^2$ , since  $w < 1$ . ■



**2.2. Instability index count.** In this section, we introduce the instability indices counting formulas, which in many cases of interest can in fact be used to accurately determine both stability and instability regimes for the waves under consideration. As we have mentioned above, this theory has been under development for some time (see [18], [19], [14], [20]), but we use a recent formulation due to Kapitula, Kevrekidis, and Sandstede (KKS) [15] (see also [16]). In fact, even the KKS index count formula is not directly applicable<sup>2</sup> to the problem of (5), which is why Kapitula and Stefanov [17] have found an approach, based on the KKS of the theory, which covers this situation. In order to simplify the exposition, we will restrict our attention to a corollary of the main result in [17]. More precisely, the stability problem in the form is considered in the form

$$(10) \quad \partial_x \mathcal{L}u = \lambda u,$$

where  $\mathcal{L}$  is a self-adjoint linear differential operator with domain  $D(\mathcal{L}) = H^s(\mathbf{R}^1)$  for some  $s$ . It is assumed that for the operator  $\mathcal{L}$ ,

- (1) there are  $n(\mathcal{L}) = N < +\infty$  negative eigenvalues<sup>3</sup> (counting multiplicity) so that each of the corresponding eigenvectors  $\{f_j\}_{j=1}^N$  belong to  $H^{1/2}(\mathbf{R}^1)$ ,
- (2) there is a  $\kappa > 0$  such that  $\sigma_{ess}(\mathcal{L}) \subset [\kappa^2, +\infty)$ , and
- (3)  $\dim[\ker(\mathcal{L})] = 1$ ,  $\ker(\mathcal{L}) = \text{span}\{\psi_0\}$ ,  $\psi_0$  is a real-valued function, and  $\psi_0 \in H^\infty(\mathbf{R}^1) \cap \dot{H}^{-1}(\mathbf{R}^1)$ .

Here,  $\dot{H}^{-1}(\mathbf{R}^1)$  is the homogeneous Sobolev space, defined via the norm

$$\|u\|_{\dot{H}^{-1}(\mathbf{R}^1)} := \left( \int_{\mathbf{R}^1} \frac{|\hat{u}(\xi)|^2}{|\xi|^2} d\xi \right)^{1/2},$$

or, equivalently,  $u = \partial_x z$  in sense of distributions, where  $z \in L^2$  and  $\|u\|_{\dot{H}^{-1}(\mathbf{R}^1)} := \|z\|_{L^2}$ .

Before we formulate the main result of [17], let us take the opportunity to define various quantities that will appear in the index count. First, denote by  $k_r$  the number of real-valued and positive eigenvalues (counting multiplicities) of (10). Let also  $k_c$  be the number of complex eigenvalues with positive real part. Note that  $k_c$  is even by symmetries, since  $\Im \mathcal{L} = 0$ . For (potentially embedded) purely imaginary eigenvalues  $\lambda \in i\mathbf{R}$ , let  $E_\lambda$  denote the corresponding eigenspace. The negative Krein index of the eigenvalue is given by<sup>4</sup>  $k_i^-(\lambda) = n(\langle L^\diamond|_{E_\lambda} u, u \rangle)$ , and the total negative Krein index is given by  $k_i^- = \sum_{\lambda \in i\mathbf{R}} k_i^-(\lambda)$ . Again, by symmetries,  $k_i$  is an even integer as well. The total Hamilton–Krein index is then defined as

$$K_{Ham} := k_r + k_c + k_i^-.$$

**Theorem 4 (Theorem 3.5 in [17]).** *For the eigenvalue problem*

$$(11) \quad \partial_x \mathcal{L}u = \lambda u, \quad u \in L^2(\mathbf{R}^1),$$

<sup>2</sup>Due to a crucial assumption for invertibility of the skew-symmetric operator  $J$ , which is not satisfied for  $\partial_x$  acting on  $\mathbf{R}^1$ .

<sup>3</sup>We will henceforth denote by  $n(M)$  the number of negative eigenvalues (counting multiplicities) of a self-adjoint operator  $M$ .

<sup>4</sup>We let  $n(S)$  denote the number of negative eigenvalues (counting multiplicity) of the self-adjoint operator  $S$ .

where the self-adjoint operator  $\mathcal{L}$  satisfies  $D(\mathcal{L}) = H^s(\mathbf{R}^1)$  for some  $s > 0$ , assume that conditions (1), (2), and (3) above hold and in addition

$$\langle \mathcal{L}^{-1} \partial_x^{-1} \psi_0, \partial_x^{-1} \psi_0 \rangle \neq 0.$$

Then, the number of solutions of (10),  $n_{\text{unstable}}(\mathcal{L})$ , with  $\lambda : \Re \lambda > 0$  satisfies<sup>5</sup>

$$(12) \quad K_{\text{Ham}} = n(\mathcal{L}) - n(\langle \mathcal{L}^{-1} \partial_x^{-1} \psi_0, \partial_x^{-1} \psi_0 \rangle).$$

Of course, our eigenvalue problem (6) does not immediately fit the form of Theorem 4. First, Theorem 4 applies for scalar-valued operators  $\mathcal{L}$ , while we need to deal with vector-valued operators. This is a minor issue, and in fact one sees easily that the arguments in [17] carry over easily in the case where  $\mathcal{L}$  is a vector-valued self-adjoint operator as well. A second, more substantive, issue is that the form of (6) is not quite that of (11). Namely, we have that the operator  $J$ , while still skew-symmetric, is not equal to  $\partial_x$ .

In order to fix that, we need to recast the eigenvalue problem (6) in a slightly different form. Indeed, letting  $\mathbf{f} = (1 - b\partial_x^2)^{-1/2} \mathbf{g}$  and taking  $(1 - b\partial_x^2)^{1/2}$  on both sides of (6), we may rewrite it as follows:

$$\partial_x \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} (1 - b\partial_x^2)^{-1/2} L (1 - b\partial_x^2)^{-1/2} \mathbf{g} = \lambda \mathbf{g}.$$

If we now introduce

$$\tilde{J} := \partial_x \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \tilde{L} := (1 - b\partial_x^2)^{-1/2} L (1 - b\partial_x^2)^{-1/2},$$

we easily see that  $\tilde{J}$  is still antisymmetric,  $\tilde{L}$  is self-adjoint, and we have managed to represent the eigenvalue problem in the form  $\tilde{J}\tilde{L}\mathbf{g} = \lambda\mathbf{g}$ . Note that the operator  $\tilde{J}$  is very similar to  $\partial_x$ , except for the action of the invertible symmetric operator  $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$  on it. It is not hard to see that the result of Theorem 4 applies to it (while it still fails the standard conditions of the KKS theory, due to the noninvertibility of  $\tilde{J}$ ). Note that one needs to replace  $\partial_x^{-1}$  by  $\tilde{J}^{-1}$  in the formula (12). Furthermore, the number of unstable modes for the two systems ( $JL$  and  $\tilde{J}\tilde{L}$ ) is clearly the same, due to the simple transformation  $(1 - b\partial_x^2)^{-1/2}$  connecting the corresponding eigenfunctions. Let us also point out that in our application, we will always have  $n(\tilde{L}) = 1$ . Note that we do not ever restrict the parameter range in order to ensure the validity of this condition. Indeed,  $n(\tilde{L}) = 1$  always holds whenever Theorem 4 is applicable to the  $\text{sech}^2$  solutions of the abc system.

Thus, if we can verify the conditions under which Theorem 4 applies, we get the stability index formula

$$(13) \quad n_{\text{unstable}}(JL) + \text{even number} = n(\tilde{L}) - n(\langle \tilde{L}^{-1} \tilde{J}^{-1} \psi_0, \tilde{J}^{-1} \psi_0 \rangle).$$

<sup>5</sup>Here  $\partial_x^{-1} \psi_0$  is any  $L^2$  function  $f$  so that  $\psi_0 = \partial_x f$  in the distributional sense.



Since by Proposition 1  $L(\frac{\varphi'}{\psi'}) = 0$ , we conclude that  $\tilde{L}[(1 - b\partial_x^2)^{1/2}(\frac{\varphi'}{\psi'})] = 0$ . It follows that  $\psi_0 = \partial_x(1 - b\partial_x^2)^{1/2}(\frac{\varphi}{\psi})$  and

$$\begin{aligned}\langle \tilde{L}^{-1}\tilde{J}^{-1}\psi_0, \tilde{J}^{-1}\psi_0 \rangle &= \left\langle L^{-1} \left[ (1 - b\partial_x^2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right] (1 - b\partial_x^2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle \\ &= \left\langle L^{-1} \left[ (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right] (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right\rangle.\end{aligned}$$

Thus, we conclude that we will have established spectral stability for (6), if we can verify the conditions (1), (2), (3) of Theorem 4 for the operator  $\tilde{L}$ ,  $n(\tilde{L}) = 1$  and

$$(14) \quad \left\langle L^{-1} \left[ (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right], (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right\rangle < 0,$$

and instability otherwise.

Concretely, we will verify the conditions on  $\tilde{L}$  in Proposition 2 below, after which we compute the quantity in (14) in Proposition 3.

**Proposition 2.** *The self-adjoint operator  $\tilde{L} = (1 - b\partial_x^2)^{-1/2}L(1 - b\partial_x^2)^{-1/2}$  satisfies*

- (1)  $\sigma_{ess.}(\tilde{L}) \subset [\kappa, \infty)$  for some positive  $\kappa$ ,
- (2)  $n(\tilde{L}) = 1$ , and
- (3)  $\text{Ker}(\tilde{L}) = \text{span}\{(1 - b\partial_x^2)^{1/2}(\frac{\varphi'}{\psi'})\}$

in the following cases:

- $a = c = -b$ ,  $b > 0$ ,  $B = \pm\sqrt{\frac{3}{3+\eta_0}}$ ,  $w = \pm\frac{3+2\eta_0}{\sqrt{3(3+\eta_0)}}$ ,  $\eta_0 \in (-\frac{9}{4}, 0)$ ,
- $a = c < 0$ ,  $b > 0$ ,  $w = 0$ ,  $B = \pm\sqrt{2}$ .

**Proposition 3.** *Regarding the instability index, we have the following.*

- For  $a = c = -b$ ,  $b > 0$ ,  $w = \pm\frac{3+2\eta_0}{\sqrt{3(3+\eta_0)}}$ ,  $B(\eta_0) = \pm\sqrt{\frac{3}{3+\eta_0}}$ , and for all  $\eta_0 \in (-\frac{9}{4}, 0)$ ,

$$\left\langle L^{-1} \left[ (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right], (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right\rangle < 0.$$

- For  $a = c < 0$ ,  $b > 0$ ,  $w = 0$ , and  $B = \pm\sqrt{2}$ ,

$$\begin{aligned}&\left\langle L^{-1} \left[ (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right], (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right\rangle \\ &= \frac{1}{3} \left( 8\sqrt{-a} \left( -\frac{9}{2} - \frac{12}{5} \frac{b}{|a|} + \frac{3}{10} \frac{b^2}{a^2} \right) + \langle (a\partial_x^2 + 1 - \varphi)^{-1}f, f \rangle \right).\end{aligned}$$

In particular,

$$\begin{aligned}&\left\langle L^{-1} \left[ (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right], (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right\rangle < 0, \quad 0 < \frac{b}{-a} < 8.00163, \\ &\left\langle L^{-1} \left[ (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right], (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right\rangle > 0, \quad \frac{b}{-a} > 8.82864.\end{aligned}$$

Theorem 2 follows by virtue of Propositions 2 and 3. Thus, it remains to prove these two.

**3. Proof of Proposition 2.** We start with the gap condition for  $\sigma_{ess.}(\tilde{L})$  stated in Proposition 2.

**3.1.  $\tilde{L}$  is strictly positive.** The idea is contained in Proposition 1. Write

$$\tilde{L} = (1 - b\partial_x^2)^{-1/2} L (1 - b\partial_x^2)^{-1/2} = (1 - b\partial_x^2)^{-1/2} L_0 (1 - b\partial_x^2)^{-1/2} + (1 - b\partial_x^2)^{-1/2} (L - L_0) (1 - b\partial_x^2)^{-1/2},$$

where  $L - L_0$  is a multiplication by smooth and decaying potential. It is also not hard to see that  $(1 - b\partial_x^2)^{-1/2}$  is given by a convolution kernel  $K : K(x) = \int_{-\infty}^{\infty} \frac{e^{2\pi i x \xi}}{\sqrt{1 + 4\pi^2 b \xi^2}} d\xi$ , which decays faster than polynomial at  $\pm\infty$ . It follows that the operator  $(1 - b\partial_x^2)^{-1/2} (L - L_0) (1 - b\partial_x^2)^{-1/2}$  is a compact operator on  $L^2(\mathbf{R}^1)$ , and hence by Weyl's theorem

$$\sigma_{ess.}(\tilde{L}) = \sigma_{ess.}((1 - b\partial_x^2)^{-1/2} L_0 (1 - b\partial_x^2)^{-1/2}) = \sigma((1 - b\partial_x^2)^{-1/2} L_0 (1 - b\partial_x^2)^{-1/2}).$$

Thus, as we have explained in the proof of Proposition 1, it will suffice to check that the matrix

$$(1 + 4\pi^2 b \xi^2)^{-1/2} L_0(\xi) (1 + 4\pi^2 b \xi^2)^{-1/2}$$

is positive definite. But since  $L_0(\xi)$  is positive definite, the result follows. Note that this shows only that  $\sigma_{ess.}(\tilde{L}) \geq 0$ . Since we need to show an actual gap between  $\sigma_{ess.}(\tilde{L})$  and zero, it suffices to observe (by the arguments in Proposition 1) that the eigenvalues of  $L_0(\xi)$  have the rate of  $O(\xi^2)$  for large  $\xi$ , which implies that the positive eigenvalues of  $(1 + 4\pi^2 b \xi^2)^{-1/2} L_0(\xi) (1 + 4\pi^2 b \xi^2)^{-1/2}$  have the rate of  $O(1)$ .

**3.2. The negative eigenvalue and the zero eigenvalue are both simple.** We now pass to the harder task of establishing the existence and simplicity of a negative eigenvalue for  $\tilde{L}$  as well as the simplicity of the zero eigenvalue. Note that as we have already observed  $L(\frac{\varphi'}{\psi'}) = 0$ . It follows that

$$\tilde{L} \left[ (1 - b\partial_x^2)^{1/2} \begin{pmatrix} \varphi' \\ \psi' \end{pmatrix} \right] = (1 - b\partial_x^2)^{-1/2} \left[ L \begin{pmatrix} \varphi' \\ \psi' \end{pmatrix} \right] = 0.$$

Thus, we have already identified one element of  $\text{Ker}(\tilde{L})$ , but it still remains to prove that  $\dim(\text{Ker}(\tilde{L})) = 1$  in addition to the existence and the simplicity of the negative eigenvalue of  $\tilde{L}$ .

Next, we find it convenient to introduce the following notation for the eigenvalues of a self-adjoint operator  $\mathcal{L}$ . Indeed, assuming that  $\mathcal{L} = \mathcal{L}^*$  is bounded from below,  $\mathcal{L} \geq -c$ , we order<sup>6</sup> the eigenvalues as follows:

$$\inf \text{spec}(\mathcal{L}) = \lambda_0(\mathcal{L}) \leq \lambda_1(\mathcal{L}) \leq \dots$$

Recall also the following max-min principle due to Courant:

$$\lambda_0(\mathcal{L}) = \inf_{\|f\|=1} \langle \mathcal{L}f, f \rangle, \quad \lambda_1(\mathcal{L}) = \sup_{g \neq 0} \inf_{\|f\|=1, f \perp g} \langle \mathcal{L}f, f \rangle, \quad \lambda_2(\mathcal{L}) = \sup_{g_1, g_2: g_1 \neq ag_2} \inf_{\|f\|=1, f \perp \text{span}[g_1, g_2]} \langle \mathcal{L}f, f \rangle.$$

<sup>6</sup>We follow the standard convention the appearance of an equality multiple times in the sequence of eigenvalues signifies that eigenvalue has the same multiplicity.

Clearly, our claims can be recast in the more compact form

$$(15) \quad \lambda_0(\tilde{L}) < 0 = \lambda_1(\tilde{L}) < \lambda_2(\tilde{L}).$$

We note that it will often be convenient to reduce matters from  $\tilde{L}$  to standard second order differential operators, like  $L$ . The following standard lemma will be useful in that regard.

**Lemma 1.** *Let  $a, c < 0$ ,  $b > 0$ , and  $w : 0 \leq |w| < \min(1, \frac{\sqrt{ac}}{|b|})$ . Then the following hold.*

- *All eigenvectors of  $L$  from (4), corresponding to nonpositive eigenvalues, belong to  $H^\infty(\mathbf{R}^1) = \cap_{l=1}^\infty H^l(\mathbf{R}^1)$ .*
- *If  $\mathcal{L}$  is any bounded from below self-adjoint operator, for which  $\lambda_0(\mathcal{L}) < 0 = \lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L})$ , and  $S$  is a bounded invertible operator, then*

$$\lambda_0(S^* \mathcal{L} S) < 0 = \lambda_1(S^* \mathcal{L} S) < \lambda_2(S^* \mathcal{L} S).$$

- *If  $L$  has the property  $\lambda_0(L) < 0 = \lambda_1(L) < \lambda_2(L)$ , then so does  $\tilde{L} = (1 - b\partial_x^2)^{-1/2} L (1 - b\partial_x^2)^{-1/2}$ . That is, (15) holds.*

**Proof of Lemma 1.** Take the eigenvector  $\mathbf{f}$  corresponding to  $-a^2$ ,  $a \geq 0$ ; i.e.,  $L\mathbf{f} = -a^2\mathbf{f}$ . As observed in the proof of Proposition 1, we can represent  $L = L_0 + \mathbf{V}$ , where  $\mathbf{V}$  is a smooth and decaying matrix potential. In addition, recall that  $L_0 \geq \kappa$ ; hence  $L_0 + a^2 \geq \kappa Id$  and is hence invertible. It follows that the eigenvalue problem at  $-a^2$  can be rewritten in the equivalent form

$$\mathbf{f} = -(L_0 + a^2)^{-1}[\mathbf{V}\mathbf{f}].$$

Clearly,  $(L_0 + a^2)^{-1} : L^2 \rightarrow H^2$ , whence we get immediately that  $f \in H^2$  if  $f \in L^2$ . Bootstrapping this argument (recall that  $\mathbf{V} \in C^\infty$ ) yields  $f \in H^4, H^6$ , etc. In the end,  $\mathbf{f} \in H^\infty$ .

Next, we have

$$\lambda_0(S^* \mathcal{L} S) = \inf_{f: \|f\|=1} \langle S^* \mathcal{L} S f, f \rangle = \inf_{f \neq 0} \frac{\langle \mathcal{L} S f, S f \rangle}{\|f\|^2} = \inf_{g \neq 0} \frac{\langle \mathcal{L} g, g \rangle}{\|S^{-1} g\|^2} < 0,$$

since  $\lambda_0(\mathcal{L}) = \inf_{g: \|g\|=1} \langle \mathcal{L} g, g \rangle < 0$ . Since  $\lambda_1(\mathcal{L}) = 0$ , it follows that there is  $h$  so that  $\inf_{g \perp h} \langle \mathcal{L} g, g \rangle \geq 0$ . Thus,

$$\lambda_1(S^* \mathcal{L} S) \geq \inf_{f \perp S^* h} \frac{\langle S^* \mathcal{L} S f, f \rangle}{\|f\|^2} = \inf_{g \perp h} \frac{\langle \mathcal{L} g, g \rangle}{\|S^{-1} g\|^2} \geq 0.$$

Since 0 is still an eigenvalue for  $\mathcal{L}$  with, say, eigenvector  $\chi$ , it follows that  $S^{-1}\chi$  is an eigenvector to  $S^* \mathcal{L} S$ , so 0 is also an eigenvalue for  $S^* \mathcal{L} S$ , and hence  $\lambda_1(S^* \mathcal{L} S) = 0$ .

Regarding  $\lambda_2(S^* \mathcal{L} S)$ , we already know that  $\lambda_2(S^* \mathcal{L} S) > \lambda_1(S^* \mathcal{L} S) = 0$ . Assuming the contrary would mean that  $\lambda_2(S^* \mathcal{L} S) = 0$ ; that is, 0 is a double eigenvalue for  $S^* \mathcal{L} S$ , say, with linearly independent eigenvectors  $f_1, f_2$ . From this and the invertibility of  $S$ , it follows that  $S^{-1}f_1, S^{-1}f_2$  are two linearly independent vectors in  $\text{Ker}(L)$ , in contradiction to the assumption that 0 is a simple eigenvalue for  $L$ .

The result regarding  $(1 - b\partial_x^2)^{-1/2} L (1 - b\partial_x^2)^{-1/2}$  follows in a similar way although clearly cannot go through the previous claim (since  $(1 - b\partial_x^2)^{-1/2}$  does not have a bounded inverse). To show that  $\lambda_0(\tilde{L}) < 0$ , take an eigenvector, say,  $g_0 : \|g_0\| = 1$ , corresponding to the negative

eigenvalue  $-a^2$  for  $L$ . Note that by the first claim, such a  $g_0$  is smooth, so in particular  $(1 - b\partial_x^2)^{1/2}g_0$  is well defined, smooth, and nonzero. We have

$$\lambda_0(\tilde{L}) \leq \frac{\langle \tilde{L}(1 - b\partial_x^2)^{1/2}g_0, (1 - b\partial_x^2)^{1/2}g_0 \rangle}{\|(1 - b\partial_x^2)^{1/2}g_0\|^2} = \frac{\langle Lg_0, g_0 \rangle}{\|(1 - b\partial_x^2)^{1/2}g_0\|^2} = -\frac{a_0^2}{\|(1 - b\partial_x^2)^{1/2}g_0\|^2} < 0.$$

Next, to show that  $\lambda_1(\mathcal{L}) \geq 0$  (the fact that 0 is an eigenvalue for  $\tilde{L}$  was established already), recall that since  $L$  has a simple negative eigenvalue, with eigenfunction  $g_0$ , we have

$$\inf_{g: g \perp g_0} \langle Lg, g \rangle = 0.$$

It follows that

$$\lambda_1(\tilde{L}) \geq \inf_{f \perp (1 - b\partial_x^2)^{-1/2}g_0} \frac{\langle \tilde{L}f, f \rangle}{\|f\|^2} = \inf_{h \perp g_0} \frac{\langle Lh, h \rangle}{\|(1 - b\partial_x^2)^{1/2}h\|^2} \geq 0.$$

Regarding the proof of  $\lambda_2(\tilde{L}) > 0$ , we start with  $\lambda_2(\tilde{L}) \geq \lambda_1(\tilde{L}) = 0$ , and we reach a contradiction as before (i.e., we generate two linearly independent vectors in  $\text{Ker}(L)$ ) if we assume that  $\lambda_2(\tilde{L}) = 0$ . ■

Using Lemma 1 allows us to reduce the proof of (15) to the proof of

$$(16) \quad \lambda_1(L) < 0 = \lambda_1(L) < \lambda_2(L),$$

on which we now concentrate.

We have

$$\begin{aligned} L &= \begin{pmatrix} 1 + a\partial_x^2 & bw\partial_x^2 + \psi - w \\ bw\partial_x^2 + \psi - w & 1 + a\partial_x^2 + \varphi \end{pmatrix} \\ &= (1 + a\partial_x^2)Id + (bw\partial_x^2 - w) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \psi \\ \psi & \varphi \end{pmatrix}. \end{aligned}$$

Introduce an orthogonal matrix  $T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$  and observe that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = T^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T.$$

It follows that

$$L = T^{-1} \left( (1 + a\partial_x^2)Id + (bw\partial_x^2 - w + \psi) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\varphi}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) T,$$

whence, by unitary equivalence, it suffices to consider the operator inside the parentheses. That is, we consider

$$(17) \quad M = \begin{pmatrix} -\partial_x^2(-a - bw) + (1 - w) + \psi + \frac{\varphi}{2} & \frac{\varphi}{2} \\ \frac{\varphi}{2} & -\partial_x^2(-a + bw) + (1 + w) - \psi + \frac{\varphi}{2} \end{pmatrix}.$$

We shall need the following lemma.

**Lemma 2.** *Let  $\alpha, \lambda > 0$  and  $Q \in \mathbf{R}^1$ . Then, the Hill operator*

$$\mathcal{L} = -\partial_x^2 + \alpha^2 - Q \operatorname{sech}^2(\lambda x) \geq 0$$

*if and only if*

$$(18) \quad \alpha^2 + \alpha\lambda \geq Q.$$

**Proof.** This is standard result, which follows from those found in the literature by a simple change of variables. First, if  $Q \leq 0$ , we see right away that  $\mathcal{L} > 0$ , and the inequality (18) is satisfied as well. So, assume  $Q > 0$ . Consider  $\mathcal{L}f = \sigma f$  and introduce  $f(x) = g(\lambda x)$ . We have (after dividing by  $\lambda^2$  and assigning  $y = \lambda x$ )

$$\left[ -\partial_{yy} + \frac{\alpha^2}{\lambda^2} - \frac{Q}{\lambda^2} \operatorname{sech}^2(y) \right] g = \frac{\sigma}{\lambda^2} g(y).$$

Recall that the negative operator  $-\partial_{yy} - Z \operatorname{sech}^2(y)$  is  $k_m = -[(Z + \frac{1}{4})^{\frac{1}{2}} - m - \frac{1}{2}]^2$  provided  $(Z + \frac{1}{4})^{\frac{1}{2}} - m - \frac{1}{2} > 0$ ,  $m = 0, 1, 2, \dots$  (see [1]). Note that  $k_0 = \inf \sigma(-\partial_{yy} - Z \operatorname{sech}^2(y))$ , and hence, to avoid negative spectrum, we need to have

$$0 \leq \frac{\alpha^2}{\lambda^2} + k_0 = \frac{\alpha^2}{\lambda^2} - \left[ \left( \frac{Q}{\lambda^2} + \frac{1}{4} \right)^{\frac{1}{2}} - \frac{1}{2} \right]^2.$$

Solving this last inequality yields (18). ■

We are now ready to proceed with the count of  $n(\tilde{L})$  in each particular case of consideration.

**Case I:  $a = c = -b$ ,  $b > 0$ .** Going back to the operator  $M$ , we can rewrite it as

$$M = S \begin{pmatrix} -\partial_x^2 + \frac{1}{b} + \frac{B+\frac{1}{2}}{b(1-w)}\varphi & \frac{\varphi}{2b\sqrt{1-w^2}} \\ \frac{\varphi}{2b\sqrt{1-w^2}} & -\partial_x^2 + \frac{1}{b} + \frac{-B+\frac{1}{2}}{b(1+w)}\varphi \end{pmatrix} S,$$

where  $S = \begin{pmatrix} \sqrt{b(1-w)} & 0 \\ 0 & \sqrt{b(1+w)} \end{pmatrix}$ . Thus, according to Lemma 1, we have reduced matters to

$$M_1 = \left( -\partial_x^2 + \frac{1}{b} \right) Id + \varphi \begin{pmatrix} \frac{B+\frac{1}{2}}{b(1-w)} & \frac{1}{2b\sqrt{1-w^2}} \\ \frac{1}{2b\sqrt{1-w^2}} & \frac{-B+\frac{1}{2}}{b(1+w)} \end{pmatrix}.$$

Diagonalizing this last symmetric matrix yields the representation

$$\begin{pmatrix} \frac{B+\frac{1}{2}}{b(1-w)} & \frac{1}{2b\sqrt{1-w^2}} \\ \frac{1}{2b\sqrt{1-w^2}} & \frac{-B+\frac{1}{2}}{b(1+w)} \end{pmatrix} = U^* \begin{pmatrix} \frac{1+2Bw+\sqrt{4B^2+4Bw+1}}{2b(1-w^2)} & 0 \\ 0 & \frac{1+2Bw-\sqrt{4B^2+4Bw+1}}{2b(1-w^2)} \end{pmatrix} U$$

for some orthogonal matrix  $U$ . Factoring out  $U^*, U$  again and using Lemma 1 once more reduces us to the operator

$$M_2 = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix},$$

which contains the following Hill operators on the main diagonal:

$$\begin{aligned} \mathcal{L}_1 &= -\partial_x^2 + \frac{1}{b} + \eta_0 \frac{1 + 2Bw + \sqrt{4B^2 + 4Bw + 1}}{2b(1-w^2)} \operatorname{sech}^2\left(\frac{x}{2\sqrt{b}}\right); \\ \mathcal{L}_2 &= -\partial_x^2 + \frac{1}{b} + \eta_0 \frac{1 + 2Bw - \sqrt{4B^2 + 4Bw + 1}}{2b(1-w^2)} \operatorname{sech}^2\left(\frac{x}{2\sqrt{b}}\right). \end{aligned}$$

Note that  $n(\tilde{L}) = n(\mathcal{L}_1) + n(\mathcal{L}_2)$ .

Using the formulas

$$B(\eta_0) = \pm \sqrt{\frac{3}{3+\eta_0}}, w(\eta_0) = \pm \frac{3+2\eta_0}{\sqrt{3(3+\eta_0)}}$$

yields

$$\begin{aligned} \mathcal{L}_1 &= -\partial_x^2 + \frac{1}{b} - \frac{3}{b} \operatorname{sech}^2\left(\frac{x}{2\sqrt{b}}\right); \\ \mathcal{L}_2 &= -\partial_x^2 + \frac{1}{b} - \frac{3\eta_0}{b(9+4\eta_0)} \operatorname{sech}^2\left(\frac{x}{2\sqrt{b}}\right). \end{aligned}$$

According to the formulas for the eigenvalues in Lemma 2 (with  $\alpha = \frac{1}{\sqrt{b}}$ ,  $\lambda = \frac{1}{2\sqrt{b}}$ ,  $Q = \frac{3}{b} > 0$ ) we have that

$$\lambda_1(\mathcal{L}_1) = \frac{\alpha^2}{\lambda^2} - \left( \sqrt{\frac{Q}{\lambda^2} + \frac{1}{4}} - \frac{3}{2} \right)^2 = 2 - (\sqrt{12.25} - 1.5)^2 = 0,$$

which indicates that  $\mathcal{L}_1$  has one negative eigenvalue, and the next one is zero, whence  $n(\mathcal{L}_1) = 1$  for all  $\eta_0 > -3$ . Thus,  $n(\tilde{L}) = 1 + n(\mathcal{L}_2)$ . It is also immediately clear that for  $\eta_0 \in (-\frac{9}{4}, 0)$ ,  $\mathcal{L}_2 > 0$  and hence  $n(\tilde{L}) = 1$ .

**Case II:  $a = c < 0$ ,  $b = d > 0$ ,  $a + b \neq 0$ .** In this case, we have  $p = \frac{c+b}{a+b} = 1$ ,  $\eta_0 = \frac{3(1-2p)}{2p} = -\frac{3}{2}$ , and thus  $w(\eta_0) = w(-3/2) = 0$ ,  $\lambda = \frac{1}{2\sqrt{-a}}$ , and  $B(\eta_0) = \pm\sqrt{2}$ . This simplifies the computations quite a bit. In fact, starting from the operator  $M$ , defined in (17), we see that it has the form

$$M = (a\partial_x^2 + 1)Id + \begin{pmatrix} B + \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -B + \frac{1}{2} \end{pmatrix} \varphi.$$

Recall that here  $B = \pm\sqrt{2}$ . Consider first  $B = \sqrt{2}$ . Diagonalizing the matrix via an orthogonal matrix  $S$  yields the representation

$$\begin{aligned} \begin{pmatrix} \sqrt{2} + \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\sqrt{2} + \frac{1}{2} \end{pmatrix} &= S^{-1} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} S, \\ S &= \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3+2\sqrt{2}} & \sqrt{3-2\sqrt{2}} \\ -\sqrt{3-2\sqrt{2}} & \sqrt{3+2\sqrt{2}} \end{pmatrix}. \end{aligned}$$

Thus, in this case, we have represented the operator  $L$  in the form

$$(19) \quad L = (ST)^* \begin{pmatrix} -a\partial_x^2 + 1 + 2\varphi & 0 \\ 0 & -a\partial_x^2 + 1 - \varphi \end{pmatrix} ST,$$

where  $S, T$  are explicit orthogonal matrices. It is now clear that since  $\eta_0 = -\frac{3}{2} < 0$ , we have that  $\varphi(x) < 0$  and hence the operator  $a\partial_x^2 + 1 - \varphi > 0$ . On the other hand,  $L_{KdV} = a\partial_x^2 + 1 + 2\varphi$  is well known to have a zero eigenvalue (with eigenfunction  $\varphi'$ ) and a unique simple negative eigenvalue.

For the case  $B = -\sqrt{2}$ , we have (19), with

$$S = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3-2\sqrt{2}} & \sqrt{3+2\sqrt{2}} \\ \sqrt{3+2\sqrt{2}} & -\sqrt{3-2\sqrt{2}} \end{pmatrix}.$$

**4. Proof of Proposition 3.** The purpose of this section is to compute the quantity appearing in (14), whose negativity will be equivalent to the stability of the waves. Thus, we need to find

$$L^{-1} \left[ (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right]$$

Here, our considerations need to be split into two cases:  $a = c = -b$ , and  $a = c < 0$ ,  $b > 0$ .

The case  $a = c = -b$  is easier to manage, since in it we have a *free parameter*  $w = w(\eta_0)$  that we can differentiate with respect to in (3). The remaining case is harder, because the parameter  $\eta_0 = -3/2$ , whence  $w = 0$  and one cannot apply the same technique.

**4.1. The case  $a = c = -b$ ,  $b > 0$ .** Taking a derivative with respect to  $w$  in (3), we find

$$L \left[ \begin{pmatrix} \partial_w \varphi \\ \partial_w \psi \end{pmatrix} \right] = (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix},$$

whence

$$L^{-1} \left[ (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right] = \begin{pmatrix} \partial_w \varphi \\ \partial_w \psi \end{pmatrix}.$$

We obtain

$$\begin{aligned} & \left\langle L^{-1} \left[ (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right], (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \right\rangle = \left\langle (1 - b\partial_x^2) \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \begin{pmatrix} \partial_w \varphi \\ \partial_w \psi \end{pmatrix} \right\rangle \\ &= \partial_w [\langle \varphi, \psi \rangle + b \langle \varphi', \psi' \rangle] = \partial_w \left[ B(\eta_0) \int \varphi(\xi)^2 + b(\varphi'(\xi))^2 d\xi \right] \\ &= B \partial_w \left[ \int_{\mathbb{R}} [\varphi^2(\xi) + b\varphi'^2(\xi)] d\xi \right] + \partial_w B \int_{\mathbb{R}} [\varphi^2(\xi) + b\varphi'^2(\xi)] d\xi \\ &= \frac{16\sqrt{b}}{5} \left[ B \frac{d\eta_0^2}{dw} + \eta_0^2 \frac{dB}{dw} \right] = \frac{16\sqrt{b}}{5} \left[ 2B + \eta_0 \frac{dB}{d\eta_0} \right] \eta_0 \frac{d\eta_0}{dw} =: d(w). \end{aligned}$$

We are now ready to compute this last expression in the cases of interest.



**4.1.1.**  $B(\eta_0) = -\sqrt{\frac{3}{3+\eta_0}}$ ,  $w = -\frac{3+2\eta_0}{\sqrt{3(3+\eta_0)}}$ . We have

$$\frac{d\eta_0}{dw} = -\frac{2\sqrt{3}(3+\eta_0)^{\frac{3}{2}}}{2\eta_0+9}, \quad \frac{dB}{d\eta_0} = \frac{\sqrt{3}}{2} \frac{1}{(3+\eta_0)^{\frac{3}{2}}},$$

and

$$d(w) = -\frac{48\sqrt{3b}}{10(3+\eta_0)^{\frac{3}{2}}}(4+\eta_0)\eta_0 \frac{d\eta_0}{dw} < 0$$

for  $-\frac{9}{4} < \eta_0 < 0$ .

**4.1.2.**  $B(\eta_0) = \sqrt{\frac{3}{3+\eta_0}}$ ,  $w = \frac{3+2\eta_0}{\sqrt{3(3+\eta_0)}}$ . We have

$$\frac{d\eta_0}{dw} = \frac{2\sqrt{3}(3+\eta_0)^{\frac{3}{2}}}{2\eta_0+9}, \quad \frac{dB}{d\eta_0} = -\frac{\sqrt{3}}{2} \frac{1}{(3+\eta_0)^{\frac{3}{2}}},$$

and hence

$$d(w) = \frac{48\sqrt{3b}}{10(3+\eta_0)^{\frac{3}{2}}}(4+\eta_0)\eta_0 \frac{d\eta_0}{dw} < 0$$

for  $-\frac{9}{4} < \eta_0 < 0$ .

**4.2. The case  $a = c < 0$ ,  $b > 0$ .** As we have discussed above, we have explicit formulas for all the quantities involved. Namely, we have  $w = 0$ ,  $\lambda = \frac{1}{2\sqrt{-a}}$ ,  $B = \pm\sqrt{2}$ . Thus,

$$\varphi(x) = -\frac{3}{2} \operatorname{sech}^2\left(\frac{x}{2\sqrt{-a}}\right).$$

**4.2.1. Case  $B = \sqrt{2}$ .** We need to compute

$$\left\langle L^{-1} \begin{pmatrix} (1-b\partial_x^2)\psi \\ (1-b\partial_x^2)\varphi \end{pmatrix}, \begin{pmatrix} (1-b\partial_x^2)\psi \\ (1-b\partial_x^2)\varphi \end{pmatrix} \right\rangle.$$

To that end, we use the representation (19). We have

$$\begin{aligned} I &= \left\langle L^{-1} \begin{pmatrix} (1-b\partial_x^2)\psi \\ (1-b\partial_x^2)\varphi \end{pmatrix}, \begin{pmatrix} (1-b\partial_x^2)\psi \\ (1-b\partial_x^2)\varphi \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} a\partial_x^2+1+2\varphi & 0 \\ 0 & a\partial_x^2+1-\varphi \end{pmatrix} \begin{bmatrix} ST\left(\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}\right)(1-b\partial_x^2)\varphi \\ ST\left(\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}\right)(1-b\partial_x^2)\varphi \end{bmatrix} \right\rangle. \end{aligned}$$

A direct computation shows that  $ST\left(\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2\sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$ , whence our index  $I$  can be computed as follows:

$$I = \frac{8}{3} \langle (a\partial_x^2+1+2\varphi)^{-1}[(1-b\partial_x^2)\varphi], (1-b\partial_x^2)\varphi \rangle + \frac{1}{3} \langle (a\partial_x^2+1-\varphi)^{-1}[(1-b\partial_x^2)\varphi], (1-b\partial_x^2)\varphi \rangle.$$

Denote  $f = (1 - b\partial_x^2)\varphi$  and

$$\begin{aligned} L_{KdV} &= a\partial_x^2 + 1 + 2\varphi, \\ L_{Hill} &= a\partial_x^2 + 1 - \varphi. \end{aligned}$$

Note that by Weyl's theorem  $\sigma_{ess.}(L_{Hill}) = [1, \infty)$ . On the other hand, by the fact that  $\varphi < 0$ , the potential  $-\varphi > 0$  and hence, by the results for absence of embedded eigenvalues,  $\sigma(L_{Hill}) = \sigma_{ess.}(L_{Hill}) = [1, \infty)$ . We now compute the index

$$I = \frac{1}{3}(8\langle L_{KdV}^{-1}f, f \rangle + \langle L_{Hill}^{-1}f, f \rangle).$$

To that end, we differentiate the equation

$$(20) \quad a\varphi'' + \varphi + \varphi^2 = 0$$

with respect to  $a$ . We get<sup>7</sup>

$$(21) \quad L_{KdV}\varphi_a = -\varphi'',$$

whence  $L_{KdV}^{-1}[\varphi''] = -\varphi_a$ . Using that  $L_{KdV}\varphi = \varphi^2 = -a\varphi'' - \varphi$  and the above relation, we obtain that

$$-\varphi = aL_{KdV}^{-1}\varphi'' + L_{KdV}^{-1}\varphi = -a\varphi_a + L_{KdV}^{-1}\varphi.$$

It follows that

$$(22) \quad L_{KdV}^{-1}\varphi = a\varphi_a - \varphi,$$

$$(23) \quad L_{KdV}^{-1}f = (a+b)\varphi_a - \varphi,$$

and

$$\langle L_{KdV}^{-1}f, f \rangle = (a+b)\langle \varphi_a, \varphi \rangle - b(a+b)\langle \varphi_a, \varphi'' \rangle - \langle \varphi, \varphi \rangle + b\langle \varphi, \varphi'' \rangle.$$

By direct computations

$$\begin{aligned} \langle \varphi_a, \varphi \rangle &= \frac{1}{2} \frac{d}{da} \int_{-\infty}^{+\infty} \varphi^2 dx = -\frac{3}{2\sqrt{-a}}, \\ \langle \varphi, \varphi'' \rangle &= - \int_{-\infty}^{+\infty} \varphi'^2 dx = -\frac{6}{5\sqrt{-a}}, \\ \langle \varphi_a, \varphi'' \rangle &= -\frac{1}{2} \frac{d}{da} \int_{-\infty}^{+\infty} \varphi'^2 dx = -\frac{3}{10|a|\sqrt{-a}}, \\ \langle \varphi, \varphi \rangle &= \frac{9}{2}\sqrt{-a} \int_{-\infty}^{+\infty} \operatorname{sech}^4(y) dy = 6\sqrt{-a}. \end{aligned}$$

As a consequence,

$$\begin{aligned} \langle L_{KdV}^{-1}f, f \rangle &= -\frac{3(a+b)}{2\sqrt{-a}} + \frac{3b(a+b)}{10|a|\sqrt{-a}} - 6\sqrt{-a} - \frac{6b}{5\sqrt{-a}} \\ &= -\frac{9}{2}\sqrt{-a} - \frac{12}{5} \frac{b}{\sqrt{-a}} + \frac{3b^2}{10|a|\sqrt{-a}} = \sqrt{-a} \left( -\frac{9}{2} - \frac{12}{5} \frac{b}{|a|} + \frac{3}{10} \frac{b^2}{a^2} \right). \end{aligned}$$

<sup>7</sup>We use the notation  $\varphi_a = \partial_a \varphi$  to denote the derivative with respect to  $a$ .

This yields the desired expression for the terms involving  $L_{KdV}^{-1}$ . We turn our attention to  $L_{Hill}^{-1}$ . The situation here is a bit trickier, since we cannot compute explicitly the quantities  $L_{Hill}^{-1}[\varphi], L_{Hill}^{-1}[\varphi'']$ , as required in the formula for  $I$ . Instead, we need to rely on estimates. To start with, observe that

$$L_{Hill}[\varphi] = a\varphi'' + \varphi - \varphi^2 = -2\varphi^2 = 2a\varphi'' + 2\varphi,$$

whence

$$(24) \quad L_{Hill}^{-1}[a\varphi'' + \varphi] = \frac{\varphi}{2}.$$

Since we need to compute  $\langle L_{Hill}^{-1}[f], f \rangle = \langle L_{Hill}^{-1}[\varphi - b\varphi''], \varphi - b\varphi'' \rangle$ , we write  $\varphi - b\varphi'' = -\frac{b}{a}(a\varphi'' + \varphi) + (1 + \frac{b}{a})\varphi$ , whence, by virtue of (24), we can reduce to the following expression:

$$\begin{aligned} \langle L_{Hill}^{-1}[\varphi - b\varphi''], \varphi - b\varphi'' \rangle &= -\frac{b}{2a} \langle \varphi, \varphi - b\varphi'' \rangle + \left(1 + \frac{b}{a}\right) \left\langle \varphi, -\frac{b}{2a}\varphi + \left(1 + \frac{b}{a}\right)L_{Hill}^{-1}[\varphi] \right\rangle \\ &= -\sqrt{-a} \left(6\frac{b}{a} + \frac{12}{5}\frac{b^2}{a^2}\right) + \left(1 + \frac{b}{a}\right)^2 \langle L_{Hill}^{-1}[\varphi], \varphi \rangle. \end{aligned}$$

Furthermore, by a simple scaling argument, it is clear that  $\langle L_{Hill}^{-1}[\varphi], \varphi \rangle = \alpha\sqrt{-a}$  for some absolute constant  $\alpha$  (independent of  $a, b$ ), that we need to compute. All in all, we can express

$$\begin{aligned} I &= \frac{\sqrt{-a}}{3} \left( -36 - \frac{96}{5} \frac{b}{|a|} + \frac{12}{5} \frac{b^2}{a^2} + \alpha + (6 - 2\alpha) \frac{b}{|a|} - \frac{12}{5} \frac{b^2}{a^2} + \alpha \frac{b^2}{a^2} \right) \\ &= \frac{\sqrt{-a}}{3} \left( \alpha \frac{b^2}{a^2} - \left(2\alpha + \frac{66}{5}\right) \frac{b}{|a|} + (\alpha - 36) \right). \end{aligned}$$

The last expression is clearly a quadratic function (which we will show has two real roots—one positive and one negative) in the variable  $z = \frac{b}{|a|} > 0$  and is hence negative (and we have instability), exactly when

$$(25) \quad \frac{b}{|a|} < \frac{5\alpha + 33 + \sqrt{3(410\alpha + 363)}}{5\alpha} := K(\alpha).$$

As we have pointed out above, it remains to find  $\alpha$ , for which it is enough to set  $a = -1$  and hence compute the expression  $\alpha = \langle (-\partial_x^2 + 1 - \varphi)^{-1}\varphi, \varphi \rangle$ , where  $\varphi = -\frac{3}{2}\operatorname{sech}^2(x/2)$ .

**Numerical computation of  $\alpha$ .** For each integer  $n$ , compute

$$L_{Hill}[\varphi^n] = -n[(n-1)\varphi^{n-2}(\varphi')^2 + \varphi^{n-1}\varphi''] + \varphi^n - \varphi^{n+1}.$$

From (20), we have the relations

$$\begin{aligned} \varphi'' &= \varphi + \varphi^2, \\ (\varphi')^2 &= \varphi^2 + \frac{2}{3}\varphi^3. \end{aligned}$$

Plugging this into the formula for  $L_{Hill}[\varphi^n]$  yields

$$(26) \quad L_{Hill}[\varphi^n] = -(n^2 - 1)\varphi^n - \left(\frac{2n^2}{3} + \frac{n}{3} + 1\right)\varphi^{n+1}.$$

From the last formula, we can derive the recurrence relationship,

$$(27) \quad L_{Hill}^{-1}[\varphi^{n+1}] = -\frac{1}{\frac{2n^2}{3} + \frac{n}{3} + 1}[(n^2 - 1)L_{Hill}^{-1}[\varphi^n] + \varphi^n],$$

which together with  $L_{Hill}^{-1}[\varphi^2] = -\frac{\varphi}{2}$  can be used to explicitly compute  $L_{Hill}^{-1}[\varphi^n]$  for any power  $n \geq 2$ . In addition, at least formally, we have  $L_{Hill}(1) = 1 - \varphi$ , whence we can compute the Casimir

$$(28) \quad L_{Hill}^{-1}[1] = 1 + L_{Hill}^{-1}[\varphi].$$

Next, we take the Legendre polynomial expansion of the function  $|z|$  in  $(-1, 1)$ . Namely, we take (for some large  $Q \sim 20$ )

$$g_Q(z) = \sum_{n=0}^Q a_{2n} P_{2n}(z), \quad a_{2n} = \frac{2n+1}{2} \int_{-1}^1 |x| P_{2n}(x) dx, \quad P_n(x) = \frac{1}{2^n n!} [(x^2 - 1)^n]^{(n)}.$$

We represent the polynomial  $g_Q$  in terms of powers:

$$g_Q(z) = \sum_{n=0}^Q c_{2n} z^{2n}.$$

As a consequence, we can write

$$-\frac{2}{3}\varphi = \operatorname{sech}^2(x/2) \sim c_0 + \sum_{n=1}^Q c_{2n} (\operatorname{sech}^2(x/2))^{2n} = c_0 + \sum_{n=1}^Q \frac{2^{2n}}{3^{2n}} c_{2n} \varphi^{2n}.$$

Thus,

$$-\frac{2}{3}L_{Hill}^{-1}[\varphi] \sim c_0 L_{Hill}^{-1}[1] + \sum_{n=1}^Q \frac{2^{2n}}{3^{2n}} c_{2n} L_{Hill}^{-1}[\varphi^{2n}].$$

Using the Casimir computation in (28), we can then express

$$L_{Hill}^{-1}[\varphi] \sim -\frac{c_0}{c_0 + 2/3} - \frac{1}{c_0 + 2/3} \sum_{n=1}^Q \frac{2^{2n}}{3^{2n}} c_{2n} L_{Hill}^{-1}[\varphi^{2n}].$$

Thus,

$$\alpha = \langle L_{Hill}^{-1}[\varphi], \varphi \rangle \sim -\frac{c_0}{c_0 + 2/3} \int_{-\infty}^{\infty} \varphi(x) dx - \frac{1}{c_0 + 2/3} \sum_{n=1}^Q \frac{2^{2n}}{3^{2n}} c_{2n} \langle L_{Hill}^{-1}[\varphi^{2n}], \varphi \rangle.$$

We have computed the right-hand side of this expression in Mathematica, with  $Q = 10, 20, 40$ . The resulting values of  $\alpha$  in these numerical runs were as follows:

$$\begin{aligned} Q = 10, & \quad \alpha = 2.673355765, \\ Q = 20, & \quad \alpha = 2.672511351, \\ Q = 40, & \quad \alpha = 2.672205451. \end{aligned}$$

Going back to the formula (25), we predict that the waves are stable for all values of  $b > 0$ ,  $a < 0$ , so that

$$0 < \frac{b}{|a|} < K(2.672205451) \sim 8.42083.$$

We would like to point out that it was important that we used the Casimir in (28). Indeed, while  $c_0$  indeed gets small, it is only of order  $10^{-2}$  when  $Q = 20$ , while the rest of the quantities are of the order  $10^{-6}$ .

**4.2.2. Case  $B = -\sqrt{2}$ .** In this case, the computation for the index is the same since

$$\begin{aligned} I &= \left\langle L^{-1} \begin{pmatrix} (1 - b\partial_x^2)\psi \\ (1 - b\partial_x^2)\varphi \end{pmatrix} \begin{pmatrix} (1 - b\partial_x^2)\psi \\ (1 - b\partial_x^2)\varphi \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} a\partial_x^2 + 1 + 2\varphi & 0 \\ 0 & a\partial_x^2 + 1 - \varphi \end{pmatrix} \left[ ST \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} (1 - b\partial_x^2)\varphi \right], ST \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} (1 - b\partial_x^2)\varphi \right\rangle \\ &= \frac{1}{3}(8\langle L_{KdV}^{-1}f, f \rangle + \langle L_{Hil}^{-1}f, f \rangle), \end{aligned}$$

where in the last line we have used that  $ST \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2\sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$  as above. The rest of the argument proceeds in exactly the same way, since the exact same quantity is being computed.

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